MA flows w/ degenerate data

Will make use of two analytic conditions (will later be shown to hold in appropriate geometric setting)

Cond. A: X^n proj., $L_1 \to X$ big and semi-ample
$L_2 \to X$ st $[L_1 + \epsilon L_2]$ semi-ample for $\epsilon > 0$ suff. small

Let $\omega_0 \in c_1(L)$ smth, semi-pos. closed (1,1)-form
$\chi \in c_1(L_2)$ smth closed (1,1) form, $\omega_t = \omega_0 + t\chi$

Assume ω_0 at worst vanishes along a proj. subvariety of X to finite order
ie $\exists$ E_0 on X an effective divisor st for any fixed Kähler metric θ

$$\omega_0 \geq C_\theta |S_{E_0}|^2_{h_{E_0}} \theta, \quad C_\theta > 0 \text{ const}, \ S_{E_0} \text{ a def. section of } E_0$$

Such a ω_0 always exists: $\omega_0 = \frac{1}{m} i\partial\bar\partial \log \sum_{j=0}^{d_m} |S_j^{(m)}|^2$

Cond B: Θ a smth vol. form on X, $E = \sum_{i=1}^{p} a_i E_i$, $F = \sum_{j=1}^{q} b_j F_j$ eff. div on X.
E_i, F_j irreducible components of E w/ simple normal crossings.
assume $a_i \geq 0$, $0 < b_j < 1$
Let Ω a semi-pos (n,n)-form st $\int \Omega > 0$ and $\Omega = |S_E|^2_{h_E} |S_F|^{-2}_{h_F} \Theta$

Cond on b_j $\Rightarrow \frac{\Omega}{\Theta} \in L^p$ for some $p > 1$
$\updownarrow$
log-terminal singularities

L_1 big & semi-ample $\Rightarrow$ by Kodaira's Lemma, $\exists$ effective $\mathbb{Q}$-divisor $\tilde E$ st $[L_1] - \epsilon[\tilde E]$ ample for suff. small $\epsilon > 0$
supp $E \cup$ supp $F \cup$ supp $E_0 \subset$ supp $\tilde E$
$\boxed{\omega_0 - \epsilon \mathrm{Ric}(h_{\tilde E}) > 0 \text{ for suff. small } \epsilon}$

Let $\omega_t = \omega_0 + t\chi$, $T_0 = \sup\{t \geq 0 \mid [L_1 + tL_2] \text{ semi-ample}\}$

Thm 3.2: X^n proj, Cond. A & B satisfied. Then for any $\varphi_0 \in PSH(X, \omega_0) \cap C^\infty(X)$, $\exists$ a unique $\varphi \in C^\infty([0,T_0) \times (X \setminus \tilde E))$ w/ $\varphi(t,\cdot) \in PSH(X, \omega_t) \cap L^\infty(X)$ for each $t \in [0, t_0)$ satisfying

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + i\partial\bar\partial\varphi)^n}{\Omega} & \text{on } [0,T_0) \times X \setminus \tilde E \\ \varphi(0,\cdot) = \varphi_0 & \text{on } X \end{cases}$$

Note: $[\omega_t]$ not nec. Kähler, Ω has zeros & poles

Rmk: can assume $\omega_t \geq \epsilon \omega_0$ for all $t \in [0,T]$, $\epsilon > 0$ suff. small dep on T

$$[\omega_t] = \frac{t}{T_0}\underbrace{[\omega_0 + T_0\chi]}_{\text{semi-pos. \& big}} + \frac{T_0 - t}{T_0}[\omega_0] \Rightarrow \exists\, \phi \text{ st } \omega_0 + T_0\chi + i\partial\bar\partial\phi \geq 0$$

Strategy: perturb eqn to make smth: $\omega_{t,s} = \omega_0 + t\chi + s\theta$

$$\Omega_{w,r} = \frac{r + |S_E|^2_{h_E}}{w + |S_F|^2_{h_F}}\,\Theta$$

$$\begin{cases} \dfrac{\partial \varphi_{s,w,r}}{\partial t} = \log \dfrac{(\omega_{s,t} + i\partial\bar\partial\varphi_{s,w,r})^n}{\Omega_{w,r}} \\ \varphi_{s,w,r}(0,\cdot) = \varphi_0 \end{cases}$$

Lemma 3.6: For any $s,w,r \in (0,1]$, there is a unique smth solution $\varphi_{s,w,r}$ on $[0,T_0) \times X$

Clear

Lemma 3.7: Let $F_{w,r} = \frac{\Omega_{w,r}}{\Theta^n}$. Then $\exists$ consts $p > 1$, $C > 0$ st for $w,r \in [0,1]$,

$$\|F_{w,r}\|_{L^p(X,\Theta^n)} \leq C$$

can take h_E st $|S_E|^2 \leq 1$

Pf: $$\int (F_{w,r})^p \Theta^n = \int (F_{w,r})^{p-1}\Omega_{w,r} = \int (F_{w,r})^{p-1}(r + |S_E|^2)(w + |S_F|^2)^{-1}\Theta$$
$$\leq C\int (F_{w,r})^{p-1}|S_F|^{-2}\Theta$$

$F_{w,r}$ has at worst poles along $\tilde{E}$, vanishing order of $|S_F|^2 < 2$

$\Rightarrow$ can get unif bd by taking $p - 1 > 0$ small enough

normal crossings $\Rightarrow$ can represent $\tilde{E}$ as $z_1 z_2 \ldots z_m = 0$ locally $U(z_1, \ldots, z_n)$

in small disc, have at worst

$$\int (F_{w,r})^{p-1}|S_F|^{-2}\Theta \leq C\int r^{2n-1}dr \cdot \frac{1}{r^{2m-2\epsilon}}\left(\frac{1}{r^a}\right)^{p-1}$$

$\therefore$ Can apply result about degen. cx MA since $F_{w,r}$ unif. bd in L^p

<u>Lemma 3.8:</u> $0 < T < T_0 \Rightarrow \exists\, C > 0$ st $\forall\, s, w, r \in (0,1]$,

$$\| \varphi_{s,r,w} \|_{L^\infty([0,T]\times X)} \le C$$

<u>Pf:</u> upper bd: Let $\alpha_{s,w,r}(t) = \frac{\int_X \Omega_{w,r}}{[\omega_{t,s}]^n}$, unif. bd'd in $[0,T]$ $\left(\frac{\int_X \Omega_{wr}}{[\omega_{t,s}]^n} = \frac{\int \frac{r + |S_E|^2}{w + |S_F|^2} \Theta}{\int (\omega_0 + t\chi + s\vartheta)^n,\ \vartheta^n = \Theta} \right)$

$$\frac{\partial}{\partial t} \int \varphi_{s,w,r} \Omega_{w,r} = \int \log \frac{(\omega_{t,s} + i\partial\bar\partial \varphi_{swr})^n}{\Omega_{wr}} \cdot \Omega_{wr}$$

$\log x \le x$

$$\cancel{\le \left(\int \Omega_{wr}\right) \cdot \log \left(\frac{\int (\omega_{ts} + i\partial\bar\partial\varphi_{swr})^n}{\alpha_{swr}(t) \int_X \Omega_{wr}} \right) + \alpha_{swr}(t) \int \Omega_{wr}}$$

$$\le \int (\omega_{ts} + i\partial\bar\partial \varphi_{swr})^n = \overset{?}{\alpha_{swr}(t)} \int \Omega_{wr} \Rightarrow \int \varphi_{swr} \Omega_{wr} \le C(T)$$

<u>Hörmander-Tian:</u> $\varphi_{swr} \in PSH(X, \omega_{swr}) \Rightarrow \exists\, \alpha, C_\alpha > 0$ st $\forall\, s, r \in (0,1]$, $t \in [0,T]$

$$\int e^{-\alpha(\varphi_{swr} - \sup_X \varphi_{swr})} \Omega_{wr} \le C_\alpha$$

$$\cancel{\int e^{-\alpha u} \frac{\Omega}{V} \ge e^{\log \int e^{-\alpha u} \frac{\Omega}{V}} \ge \int -\alpha u \frac{\Omega}{V}}$$

$\Rightarrow$ by <u>Jensen's Ineq:</u> $\sup_X \varphi_{swr} - \int_X \varphi_{swr} \Omega_{wr} \le C$, $C > 0$

$\Rightarrow \sup \varphi_{swr} \le C$.

<u>lower bd:</u> Must modify std max. principle arg to allow wr arbitrarily small.

Let $\theta = \delta\omega_0$, $\delta > 0$ st $2\theta \le \omega_t$ for all $t \in [0,T]$

will still have $\theta - \epsilon Ric(h_{\tilde E}) > 0$ for suff. small ϵ

M-A eqns $(\theta + i\partial\bar\partial \phi_{wr})^n = C_{wr} \Omega_{wr}$, where $[\theta]^n = C_{wr} \int \Omega_{wr}$, $\sup \phi = 0$

By Lemma 3.7, $\frac{C_{wr}\Omega_{wr}}{\theta^n}$ unif. bd'd in L^p

So by EGZ, $\phi_{wr} \in C^\infty(X \setminus \tilde E)$, and $\exists\, C > 0$ st $\forall\, w, r \in [0,1]$, $\|\phi_{wr}\|_{L^\infty} \le C$.

Now let $\psi_{swr}(t) = \varphi_{swr}(t, \cdot) - \phi_{wr}$

$$\frac{\partial}{\partial t} \psi_{swr} = \log \frac{(\omega_{ts} + i\partial\bar\partial \varphi_{swr})^n}{\Omega_{wr}} = \log \frac{(\omega_{ts} + i\partial\bar\partial \varphi_{swr})^n}{(\theta + i\partial\bar\partial \phi_{wr})^n} + \log C_{wr}$$

Apply max principle to $H = \psi_{swr} - \epsilon \log |S_{\tilde E}|^2_{h_{\tilde E}}$, $\quad i\partial\bar\partial H = i\partial\bar\partial \varphi - i\partial\bar\partial \phi + \epsilon \,\mathrm{Ric}(h_{\tilde E})$

Min achived in $X \setminus \tilde E$ since $H \to \infty$ near $\tilde E$.

$$\frac{\partial}{\partial t} H = \log \frac{(\theta + i\partial\bar\partial \phi_{wr} + (\omega_t - \theta + s\vartheta - \epsilon \,\mathrm{Ric}(h_{\tilde E})) + i\partial\bar\partial H)^n}{(\theta + i\partial\bar\partial \phi_{wr})^n} + \log C_{wr}$$

$$\geq \log \frac{(\theta + i\partial\bar\partial \phi_{wr} + (\theta - \epsilon \,\mathrm{Ric}(h_{\tilde E})) + i\partial\bar\partial H)^n}{(\theta + i\partial\bar\partial \phi_{wr})^n} + \log C_{wr}$$

$$\geq \log \frac{(\theta + i\partial\bar\partial \phi_{wr} + i\partial\bar\partial H)^n}{(\theta + i\partial\bar\partial \phi_{wr})^n} + \log C_{wr}$$

$$\geq \log C_{wr} \quad \text{at a new min of } H$$

$\Rightarrow H \geq -C$, C indep of ϵ, for ϵ suff. small $\forall\, s, w, r \in (0,1]$, $t \in [0,T]$

$\Rightarrow \varphi_{swr}(t,\cdot) \geq \phi_{wr} + \epsilon \log |S_{\tilde E}|^2 - C$

Now let $\epsilon \to 0$. $\square$

Volume and C^2 estimates.

Lemma 3.9: For any $T \in (0, T_0)$, $\exists\, C, \alpha > 0$ st $\forall\, t \in [0,T]$ and $swr \in (0,1]$,

$$\left|\frac{\partial}{\partial t} \varphi_{swr}\right| \leq C + \log |S_{\tilde E}|^{-2\alpha}$$

Pf: Max principle applied to $H^+ = \dot\varphi_{swr} - A^2 \varphi_{swr} + A \log |S_{\tilde E}|^2$

$H^- = \dot\varphi_{swr} + A^2 \varphi_{swr} - A \log |S_{\tilde E}|^2$

sim. to before

add complication of $\log |S_{\tilde E}|^2$, remove t's, as in $t\dot\varphi$

~~Lemma 3.10: $\forall\, T \in (0, T_0)$, $\exists\, C, \alpha > 0$~~

Lemma 3.10: $\forall T \in (0,T_0)$, $\exists C, \alpha > 0$ st $\forall t \in [0,T]$, $s,w \in (0,1]$

$$|\operatorname{tr}_{\vartheta}(\omega_{swr})| \leq C|S_{\tilde{E}}|_{h_{\tilde{E}}}^{-2\alpha}$$

Pf: Sim to C^2 estimate before, apply max principle to $H = \log \operatorname{tr}_{\vartheta}(\omega_{swr}) - A^2\varphi_{swr} + A\log|S_{\tilde{E}}|^2$

$$\left(\tfrac{\partial}{\partial t} - \Delta_{swr}\right)\log \operatorname{tr}_{\vartheta}(\omega_{swr}) \leq C \operatorname{tr}_{\omega_{swr}}\vartheta + \frac{\operatorname{tr}_{\omega_{swr}}(\operatorname{Ric}\Omega_{wr})}{\operatorname{tr}_{\omega_{swr}}\vartheta} + C$$

get: at a max of H: $H(t_0, z_0)$, $z_0 \in X \setminus \tilde{E}$,

$$\operatorname{tr}_{\omega_{swr}}\vartheta \leq A^{-1}\frac{\operatorname{tr}_{\omega_{swr}}(A\operatorname{Ric}(\Omega_{wr}))}{\operatorname{tr}_{\omega_{swr}}(\vartheta)} - A\log\frac{\omega_{swr}^n}{\Omega_{wr}} + C \leq C|S_{\tilde{E}}|^{-2\alpha}$$

$$\Rightarrow \operatorname{tr}_{\vartheta}(\omega_{swr}) \leq C|S_{\tilde{E}}|^{-2\alpha} \qquad \square$$

Also have $\forall T \in (0,T_0)$, $K \subset\subset X \setminus \tilde{E}$, $k \geq 0$, $\exists C_{k,K,T}$ st $\|\varphi_{swr}\|_{C^k([0,T]\times K)} \leq C_{k,K,T}$
by Schauder theory

Now must let $r, s, w \to 0$

Lemma 3.11: (Monotonicity)

1: $0 < r_1 \leq r_2 \leq 1$, $s,w \in (0,1]$ $\Rightarrow$ $\varphi_{s,w,r_1} \geq \varphi_{swr_2}$

2. $0 < w_1 \leq w_2 \leq 1$, $s,r \in (0,1]$ $\Rightarrow$ $\varphi_{s,w_1,r} \leq \varphi_{s,w_2,r}$

3. $0 < s_1 \leq s_2 \leq 1$, $r,w \in (0,1]$ $\Rightarrow$ $\varphi_{s_1wr} \leq \varphi_{s_2wr}$

Pf: Take $\frac{\partial}{\partial r_1}$ etc of M-A flow, max. principle

Limit: Fix T, let $\varphi_{s,w}(t,\cdot) = \left(\limsup_{r\to 0}\varphi_{swr}(t,\cdot)\right)^*$, where $f^*(z) = \lim_{\delta\to 0}\sup_{B_\delta(z)} f(\cdot)$

Then $\varphi_{sw} \in PSH(X, \omega_{ts}) \cap L^\infty(X) \cap C^\infty(X\setminus\tilde{E})$ and $\varphi_{swr} \to \varphi_{sw}$ on $X \setminus \tilde{E}$

~~by Lemma 3.8 and Prop 3.7~~

Now take ordinary limits in w, then s, $\varphi_s \to \varphi$

Cor: φ satisfies $\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{(\omega_t + i\partial\bar\partial\varphi)^n}{\Omega} & \text{on } [0,T]\times(X \setminus \hat{E}) \\ \varphi(0,\cdot) = \varphi_0 & \text{on } X \end{cases}$

Uniqueness harder: can use ~~max~~ min principle on $\psi_{s,\epsilon} = \varphi_s - \varphi' - \epsilon s \log|S_{\tilde{E}}|^2$, φ' another solution

$\epsilon \to 0 \Rightarrow \varphi' \le \varphi_s$

$s \to 0 \Rightarrow \varphi' \le \varphi$

$$\frac{\partial}{\partial t}\psi_{s,\epsilon} = \log \frac{(\omega_t + i\partial\bar\partial\varphi' + s(\vartheta - \epsilon Ric(h_{\tilde{E}})) + i\partial\bar\partial\psi_{s\epsilon})^n}{(\omega_t + i\partial\bar\partial\varphi')^n}$$

But showing other inequality harder

Solution vary equation by one more parameter, $\omega_{ts}^{(\delta)} = (1-\delta)\omega_0 + t\chi + s\vartheta = \omega_{ts} - \delta\omega_0$

$$\begin{cases} \frac{\partial}{\partial t}\varphi^{(\delta)}_{swr} = \log \frac{(\omega_{ts}^{(\delta)} + i\partial\bar\partial\varphi^{(\delta)}_{swr})^n}{\Omega_{wr}} \\ \varphi^{(\delta)}_{swr}\big|_{t=0} = (1-\delta)\varphi_0 \end{cases}$$

$v_\delta = \varphi' - \varphi^{(\delta)} - \delta^2 \log|S_{\tilde{E}}|^2$

$$\frac{\partial}{\partial t} v_\delta = \log \frac{(\omega_t^{(\delta)} + i\partial\bar\partial\varphi^{(\delta)} + \delta(\omega_0 - \delta Ric(h_{\tilde{E}})) + i\partial\bar\partial v_\delta)^n}{(\omega_t^{(\delta)} + i\partial\bar\partial\varphi^{(\delta)})^n}$$

Use max principle to show $C\log|S_{\tilde{E}}|^2 - C \le \frac{\partial}{\partial\delta}\varphi^{(\delta)}_{swr} \le C$

$\Rightarrow$ have Lipschitz continuity of $\varphi^{(\delta)}$ in δ,

$\Rightarrow$ can take $\delta \to 0$ limit.

Rmk: Rough & Degen.: mostly a combination of techniques w/ additional limit parameter

$(\omega_0 + s\vartheta + i\partial\bar\partial\varphi_{(0,s)})^n = F_s e^{\varphi_{(0,s)}}$ (H)

$(\omega_0 + s\vartheta + i\partial\bar\partial\hat\varphi_{(s,\gamma)})^n = F_{s\gamma} e^{\hat\varphi_{(s,\gamma)}}$ (H)

KRF on varieties w/ log terminal singularities

X $\mathbb{Q}$-factorial proj. variety w/ at worst log terminal sing

ie $\pi : \tilde{X} \to X$ a res. $\tilde{E} = Exc(\pi)$

$K_{\tilde{X}} = \pi^* K_X + \sum a_i E_i$, E_i irred. components of $Exc(\pi)$

log terminal $\Rightarrow a_i > -1$

K_X $\mathbb{Q}$-Cartier $\Rightarrow \exists\, m \in \mathbb{Z}$ st mK_X is Cartier

Def: Ω is a smth vol. form on X if Ω is a smth (n,n)-form st in local coords, $\Omega = f_U (\alpha \wedge \bar{\alpha})^{\frac{1}{m}}$, f_U smth pos fn on U, α a local gen. of mK_X

$\chi = i\partial\bar\partial \log \Omega$ a well-def. smth closed $(1,1)$-form, $\chi \in [K_X]$

$\chi = i\partial\bar\partial \log h_\Omega$, $h_\Omega = \Omega^{-1}$ a smth herm. metric on K_X

$\pi^*\Omega$ a non-neg. (n,n)-form on $\tilde{X}$ w/ zeros or poles along E_i of order $|a_i|$

$\pi^*\chi = Ric(\pi^* h_\Omega)$

H an ample divisor, $\omega = Ric(h_H) = -i\partial\bar\partial \log h_H > 0$

$\frac{\partial [\omega]}{\partial t} = [\chi] = [K_X] \Rightarrow [\omega] = [\omega_0] + t[\chi]$

Lifted M-A flow: $\begin{cases} \frac{\partial \tilde\varphi}{\partial t} = \log \frac{(\pi^*\omega_t + i\partial\bar\partial\tilde\varphi)^n}{\pi^*\Omega} & \text{on } \tilde{X} \setminus \tilde{E} \\ \tilde\varphi(0,\cdot) = \pi^*\varphi_0 & \end{cases}$ $\Rightarrow \tilde\varphi \to \varphi$ on X_{reg}

H $\quad (K_{\tilde{X}} - \tilde{E})$

until $T_0 = \sup\{t > 0 \mid H + tK_X \text{ is nef}\} \longleftrightarrow T_0 = \sup\{t > 0 \mid [L_1 + tL_2] \text{ semi-ample}\}$

Then $\pi^*\omega_0 \longleftrightarrow \omega_0$

$\pi^*\chi \longleftrightarrow \chi$ satisfy Cond. A & B from before

$\pi^*\Omega \longleftrightarrow \Omega$

$K_{\tilde{X}} = \pi^* K_X + \tilde{E} \Rightarrow \pi^*(\cancel{\omega_0 +}\, \chi) + Ric(h_{\tilde{E}})$

Extending KRF through singularities by divisorial contraction

$\pi : X \to Y$ a divisorial contraction, fibers of π connected

$\Rightarrow Y$ again a $\mathbb{Q}$-factorial proj. variety w/ at worst log terminal sing. if X is.

Prop 5.1: Let Ω_Y a smth vol. form on Y, $H_Y = \pi_*(H_0 + T_0 K_X)$

Then for some $p > 1$, $(\pi^{-1})^* \tilde{\omega}_{T_0} \in \{(\omega + i\partial\bar\partial\varphi) \mid [\omega] \in H_Y,\ \varphi \in PSH_p(X, \omega, \Omega)\}$

Pf: $\tilde{\omega}_{T_0}$ has bd'd local potential & rest. of $\tilde{\omega}_{T_0}$ has const. local pot. along each fiber of π $\to$ $\tilde{\omega}_{T_0}$ descends to Y ($(\pi^{-1})^*\tilde{\omega}_{T_0}$ well-def, bd'd local pot)

Let $F = \dfrac{(\tilde{\omega}_{T_0})^n}{\Omega_Y}$ suff. to show $F \in L^p$, some $p > 1$

$$\int_Y F^p \Omega_Y = \int_Y \left(\frac{\tilde{\omega}^n_{T_0}}{\Omega_Y}\right)^{p-1} \tilde{\omega}^n_{T_0} = \int_X \left(\frac{\tilde{\omega}^n_{T_0}}{\pi^*\Omega_Y}\right)^{p-1} \tilde{\omega}^n_{T_0} \le C \int \left(\frac{\Omega}{\Omega_Y}\right)^{p-1} \Omega$$

where $\frac{\Omega}{\Omega_Y}$ has at worst poles $\Rightarrow$ integrable for $p-1 > 0$ suff. small. $\square$

Similar pf ~~to~~ for extension through flips

$$\begin{tikzcd} X \arrow[rr, dashed, "(\pi^+)^{-1}\circ\pi"] \arrow[dr, "\pi"'] & & X^+ \arrow[dl, "\pi^+"] \\ & Y & \end{tikzcd}$$

KRF & MMP: X not minimal $\Rightarrow$ run KRF until T_0

$H + T_0 K_X$ nef, flow exists uniquely ~~up to~~ for $t \in [0, T_0)$

$H + T_0 K_X$ semi-ample by base-pt free thm $\Rightarrow R(X, H + T_0 K_X) = \bigoplus_{m=0}^{\infty} H^0(X, m(H + T_0 K_X))$ finitely generated

generating sections

$x \mapsto [\sigma_0(x), \ldots, \sigma_N(x)]$

- If $H + T_0 K_X$ big, $R(X, H + T_0 K_X)$ induces $\pi : X \to Y$

for generic H, π contracts one extremal ray of $\overline{NE}(X)$

Cases

1. π is a divisorial contraction, ie $\mathrm{Exc}(\pi)$ is a div. whose image has codim at least two. Then Y is $\mathbb{Q}$-factorial & has log-terminal sing
2. π is a small contraction, ie $\mathrm{Exc}(\pi)$ has codim ≥ 2 then Y ~~can~~ has bad sing, K_Y no longer $\mathbb{Q}$-Cartier (instead, do a flip to X^+, continue)

- $H + T_0 K_X$ not big $\Rightarrow X$ a Mori fiber space admitting ~~a~~ Fano fibration over Y